A New Facet Generating Procedure for the Stable Set Polytope

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Abstract

We introduce a new facet-generating procedure for the stable set polytope, based on replacing $(k-1)$-cliques with certain k-partite graphs, which subsumes previous procedures based on replacing vertices with stars, and thus also many others in the literature. It can be used to generate new classes of facet-defining inequalities.

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1 Introduction

Let $G = (V, E)$ be a simple, finite, undirected graph. A subset $S \subseteq V$ is a *stable set* if no two vertices of S are adjacent. The *stable set polytope* of G is the convex hull of the incidence vectors of all the stable sets of G.

$$\mathrm{STAB}(G) = \mathrm{conv}\{x \in \{0,1\}^n : x_v + x_u \leq 1, \forall \{v, u\} \in E\}$$

The facial structure of STAB(G) has been extensively studied, not only because stable set problems have applications in various fields, but also because they model other important combinatorial problems, such as set packing, set partitioning [5] and vertex coloring [3].

In the 1970's it was shown that facet-defining inequalities for STAB(H), the stable set polytope of a vertex induced subgraph H of G, can be transformed into facet-defining inequalities for STAB(G) [6,5]. Since then, other procedures based on graph-theoretical transformations, such as subdividing edges [7], subdividing stars [1], replacing vertices with stars [2] and replacing edges with *gears* [4], have been described.

In this paper, we introduce a new facet-generating procedure, based on replacing $(k-1)$-cliques with certain k-partite graphs, which subsumes previous procedures based on replacing vertices with stars, and thus also many others, including subdividing edges and subdividing stars (see [2] for details).

The procedure comes naturally from the fact that, as we shall prove, certain faces of STAB(G) are affinely isomorphic to the stable set polytopes of other smaller graphs. We can then use an extended version (introduced in Section 2) of the sequential lifting procedure [6,5] to transform facets of these faces into facets of STAB(G).

2 Preliminaries

Consider a polytope P and one of its faces F. Facet-defining inequalities for F can be transformed into facet-defining inequalities for P in the following way. First, we find a sequence of polytopes $F_1, \ldots, F_k$ such that $F_1 = F$, $F_k = P$ and F_i is a facet of F_{i+1}, for $i = 1, \ldots, k-1$; then we repeatedly apply the following theorem. Note that different sequences may yield different inequalities.

Theorem 2.1 *Let P be a convex polytope and S a finite set such that $P = \mathrm{conv}(S)$. If $cx \leq d$ is facet-defining for P and $\pi x \leq \pi^*$ is facet-defining for $\{x \in P : cx = d\}$, then $\pi x + \alpha(cx - d) \leq \pi^*$ is facet-defining for P, where $\alpha = \max\left\{\frac{\pi^* - \pi x}{cx - d} : x \in S,\ cx < d\right\}$.*

Proof. Let $\bar{x} \in S$. We know that $c\bar{x} \leq d$. If $c\bar{x} = d$, then $\bar{x} \in \{x \in P : cx = d\}$, and $\pi\bar{x} + \alpha(c\bar{x} - d) = \pi\bar{x} \leq \pi^*$. If $c\bar{x} < d$, then $\alpha \geq \frac{\pi^* - \pi\bar{x}}{c\bar{x} - d}$, and $\pi\bar{x} + \alpha(c\bar{x} - d) \leq \pi^*$. Therefore, the inequality is valid for P. Take a set $\{x^1, \ldots, x^k\} \subseteq \{x \in P : cx = d, \pi x = \pi^*\}$ of $\dim(P) - 1$ affinely independent points, and take $x^0 \in \{x \in S : cx < d\}$ such that $\alpha = \frac{\pi^* - \pi x^0}{cx^0 - d}$. We know that x^0 is not an affine combination of $x^1, \ldots, x^k$. Therefore, $\{x^0, \ldots, x^k\} \subseteq \{x \in P : \pi x + \alpha(cx - d) = \pi^*\}$ contains $\dim(P)$ affinely independent points. □

Two polytopes $P_1 \subseteq \mathbb{R}^r$ and $P_2 \subseteq \mathbb{R}^s$ are *affinely isomorphic*, denoted by $P_1 \cong P_2$, if there is an affine map $f : \mathbb{R}^r \to \mathbb{R}^s$ that is a bijection between the points of the two polytopes. Facet-defining inequalities for P_1 can be trivially transformed into facet-defining inequalities for P_2.

In the following section, we also need some new concepts related to hypergraphs. We say that two hyperedges are *strongly adjacent* if both have same size k and share exactly $k - 1$ vertices. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a *strong hypertree* either if $\mathcal{E} = \{V\}$ or if there is a leaf $v \in V$ incident to a hyperedge $e \in \mathcal{E}$ such that e is strongly adjacent to some other hyperedge of $\mathcal{H}$ and such that $(V \setminus \{v\}, \mathcal{E} \setminus \{e\})$ is also a strong hypertree. A *strong hyperpath* is a strong hypertree with exactly two leaves. We say that the strong hyperpath *connects* the leaves. Similarly to ordinary trees, it can be shown that every strong hypertree with n vertices is a k-uniform hypergraph with $n - k + 1$ hyperedges, that its incidence matrix has full rank, and that there is a strong hyperpath connecting each pair of non-adjacent vertices.

3 The procedure

Let $\mathcal{Q}$ be the set of maximal cliques of G, and $\mathcal{C}(G) = (V, \mathcal{Q})$ be the clique-hypergraph of G. Let $T = (V_T, \mathcal{Q}_T) \subseteq \mathcal{C}(G)$ be a k-uniform strong hypertree such that the subgraph of G induced by V_T is k-partite with vertex classes $V_1, \ldots, V_k$, and such that no vertex in $V_0 := V \setminus V_T$ has neighbors in all classes $V_1, \ldots, V_k$. Consider the face $F_T := \{x \in \text{STAB}(G) : x_Q = 1, \forall Q \in \mathcal{Q}_T\}$, where x_R is a shorthand for $\sum_{r \in R} x_r$, for $R \subseteq V$. We have the two following lemmas.

Lemma 3.1 $\dim(F_T) = |V| - |\mathcal{Q}_T|$.

Proof. Because the incidence matrix of T has rank $|\mathcal{Q}_T|$, it follows that $\dim(F_T) \leq |V| - |\mathcal{Q}_T|$. Take $i \in \{1, \ldots, k\}$ and let x^i be the incidence vector of V_i. Clearly, $x^i \in \text{STAB}(G)$. We prove that $x^i_Q = 1$, for all $Q \in \mathcal{Q}_T$. Suppose there exists Q such that $x^i_Q = 0$. By the pigeonhole principle, two vertices of Q belong to some other class V_j, but this contradicts the fact that V_j is a stable set. Therefore $x^i \in F_T$. Now take $v \in V_0$. There exists $i \in \{1, \ldots, k\}$ such that v is not adjacent to any vertex in V_i. Let $y^v = x^i + e_v$. It is easy to see that $y^v \in F_T$. The points $\{x^i\}_{i=1}^k \cup \{y^v\}_{v \in V_0}$ are affinely independent. This proves that $\dim(F_T) \geq |V_0| + k - 1 = |V| - |\mathcal{Q}_T|$. □

Lemma 3.2 *If $x \in F_T$ then $x_u = x_v$, $\forall u, v \in V_i$, $\forall i \in \{1, \ldots, k\}$.*

Proof. There is a strong hyperpath $Q_1, \ldots, Q_p$ in T connecting u, v. We prove the result by induction on p. If $p = 2$, then $x_{Q_1} - x_{Q_2} = x_u - x_v = 0$. If $p > 2$ then there exists $w \in V_i \cap Q_2$ such that $w \neq u, v$. As Q_1, Q_2 is a strong hyperpath with 2 hyperedges connecting u, w, we have $x_u = x_w$. Let $s = \max\{j : w \in Q_j\}$. Then $Q_s, \ldots, Q_p$ is a strong hyperpath with less than p hyperedges which connects two vertices of V_i. By inductive hypothesis, $x_w = x_v$. Therefore, $x_u = x_v$. □

Thus, each class $V_1, \ldots, V_k$ can be seen as a single vertex. Furthermore, given $S \subseteq V$ such that the incidence vector of S belongs to F_T, we can see that $V_k \subseteq S$ if and only if $\left(\bigcup_{i=1}^{k-1} V_i\right) \cap S = \emptyset$. This leads to the following construction: Let G_T be the graph obtained from G by removing the vertices of V_k and by contracting V_i into a vertex $v_i \in V_i$, for $i = 1, \ldots, k-1$.

Lemma 3.3 *If $N_G(V_k) \cap V_0 = \emptyset$ then $F_T \cong \mathrm{STAB}(G_T)$.*

Proof. $\mathrm{STAB}(G_T) \to F_T$: Take $y \in \mathrm{STAB}(G_T)$. For each $u \in V$, set $x_u = y_u$ if $u \in V_0$; $x_u = y_{v_i}$ if $u \in V_i, i \in \{1, \ldots, k-1\}$; and $x_u = 1 - \sum_{i=1}^{k-1} y_{v_i}$ if $u \in V_k$. We prove that $x \in F_T$. Let $Q \in \mathcal{Q}_T$. As Q contains exactly one vertex of each $V_1, \ldots, V_k$, we have $x_Q = \sum_{i=1}^{k-1} y_{v_i} + (1 - \sum_{i=1}^{k-1} y_{v_i}) = 1$. Take $\{a, b\} \in E$. It is straightforward to check that $x_a + x_b \leq 1$. Therefore $x \in F_T$.

$F_T \to \mathrm{STAB}(G_T)$: Take $x \in F_T$. For each $v \in V_0$, set $y_v = x_v$, and for each $i \in \{1, \ldots, k-1\}$, set $y_{v_i} = x_{v_i}$. Take $\{a, b\} \in E(G_T)$. It is straightforward to check that $y_a + y_b \leq 1$. Therefore $y \in \mathrm{STAB}(G_T)$. □

We can now use the procedure outlined in Section 2 to transform facet-defining inequalities for $\mathrm{STAB}(G_T)$ into facet-defining inequalities for $\mathrm{STAB}(G)$.

Theorem 3.4 *Suppose $N_G(V_k) \cap V_0 = \emptyset$. Let $Q_1, \ldots, Q_r$ be an ordering of $\mathcal{Q}_T$ such that the hypergraph induced by $Q_s, \ldots, Q_r$ is also a strong hypertree, for all $s \leq r$. If $\sum_{v \in V_0} \pi_v x_v + \sum_{i=1}^{k-1} \pi_{v_i} x_{v_i} \leq \pi^*$ is facet-defining for $\mathrm{STAB}(G_T)$, then $\sum_{v \in V_0} \pi_v x_v + \sum_{i=1}^{k-1} \pi_{v_i} x_{v_i} + \sum_{i=1}^{r} \alpha_i (x_{Q_i} - 1) \leq \pi^*$ is facet-defining for*

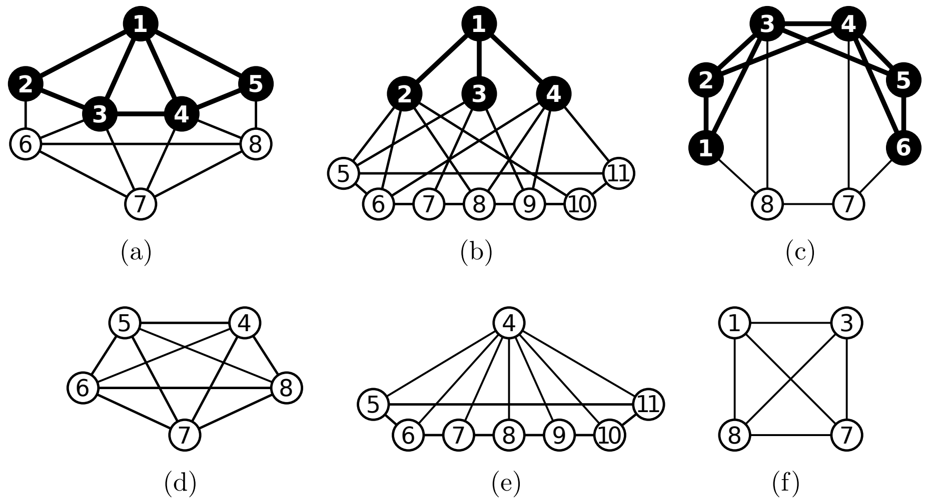

Fig. 1. Examples of Theorem 3.4

STAB(G), *where, for each* $t \in \{1, \ldots, r\}$,

$$\alpha_t := \max \left\{ \sum_{v \in V_0} \pi_v x_v + \sum_{i=1}^{k-1} \pi_{v_i} x_{v_i} + \sum_{i=1}^{t-1} \alpha_i (x_{Q_i} - 1) : x \in P_t \right\} - \pi^*$$

$$P_t := \left\{ x \in \mathrm{STAB}(G) : \begin{array}{l} x_{Q_t} = 0 \\ x_{Q_i} = 1, \ \textit{for } i = t+1, \ldots, r \end{array} \right\}.$$

Proof. For each $t \in \{0, \ldots, r\}$, let

$$F_t := \{x \in \mathrm{STAB}(G) : x_{Q_i} = 1, \ \text{for } i = t+1, \ldots, r\}$$
$$f_t(x) := \sum_{v \in V_0} \pi_v x_v + \sum_{i=1}^{k-1} \pi_{v_i} x_{v_i} + \sum_{i=1}^{t} \alpha_i (x_{Q_i} - 1)$$

We prove by induction that $f_t(x) \leq \pi^*$ is facet-defining for F_t. By Lemma 3.3, $F_0 \cong \mathrm{STAB}(G_T)$, and $f_0(x) \leq \pi^*$ is facet-defining for F_0. Take $t \in \{1, \ldots, r\}$. By Lemma 3.1, $\dim(F_{t-1}) = \dim(F_t) - 1$, and because $F_{t-1} = \{x \in F_t : x_{Q_t} = 1\}$, we know that $x_{Q_t} \leq 1$ is facet-defining for F_t. By the inductive hypothesis, $f_{t-1}(x) \leq \pi^*$ is facet-defining for F_{t-1}. We can use Theorem 2.1 to conclude that $f_{t-1}(x) + \alpha_t(x_{Q_t} - 1) = f_t(x) \leq \pi^*$ is facet-defining for F_t. □

Example 3.5 Let G be the graph of Figure 1(a), and let T be the strong hypertree in bold. We have $V_0 = \{6, 7, 8\}, V_1 = \{2, 4\}, V_2 = \{3, 5\}, V_3 = \{1\}$. The graph G_T is shown in Figure 1(d). The clique inequality $x_4 + x_5 +$

$x_6 + x_7 + x_8 \leq 1$ is facet-defining for STAB(G_T). Let $Q_1 = \{1, 2, 3\}, Q_2 = \{1, 3, 4\}, Q_3 = \{1, 4, 5\}$. Applying Theorem 3.4, we obtain the facet-defining inequality $2x_1 + x_2 + 2x_3 + 2x_4 + x_5 + x_6 + x_7 + x_8 \leq 3$.

Example 3.6 We illustrate how Theorem 3.4 subsumes previous procedures with an example from [2]. Let G be the graph of Figure 1(b), and let T be the star in bold. Note that the conditions of the first part of Theorem 3.6 of [2] are not satisfied. The graph G_T is shown in Figure 1(e). The wheel inequality $3x_4 + \sum_{i=5}^{11} x_i \leq 3$ is facet-defining for STAB(G_T). Using different orderings of Q_T, we obtain the three following facet-defining inequalities for STAB(G):

$$2x_1 + x_2 + 2x_3 + 2x_4 + \sum_{i=5}^{11} x_i \leq 5$$

$$2x_1 + 2x_2 + x_3 + 2x_4 + \sum_{i=5}^{11} x_i \leq 5$$

$$2x_1 + 2x_2 + 2x_3 + x_4 + \sum_{i=5}^{11} x_i \leq 5$$

Example 3.7 Finally, we illustrate how Theorem 3.4 can be used to generate facet-defining inequalities for antiwebs and other similar graphs. Let G be the graph of Figure 1(c), and let T be the strong hypertree in bold. The graph G_T is shown in Figure 1(f). The clique inequality $x_1 + x_3 + x_7 + x_8 \leq 1$ is facet-defining for STAB(G_T). Applying Theorem 3.4, we obtain the facet-defining inequality $x_1 + x_2 + 2x_3 + 2x_4 + x_5 + x_6 + x_7 + x_8 \leq 3$.

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